

A remark on 2-adjacency relation between Montesinos knots and 2-bridge knots

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Abstract

We show that for every 2-bridge knot K , there is an infinite family of Montesinos knots $\{K_q\}$ such that each K_q is 2-adjacent to K . We also give a new example of 2-adjacency relation between 2-bridge knots.

1 Introduction

Let K and K' be knots in S^3 . K is called n -adjacent to K' for some $n \in \mathbb{N}$, if K admits a diagram containing n (generalized) crossings such that changing any non-empty subset of them yields a diagram of K' ([1], [5]). In [9], [11], and [12], the author studied n -adjacency relation between 2-bridge knots for $n = 2, 3$ using Dehn surgery techniques in detail. Note that 2-adjacency relation between Montesinos knots is still not determined in general.

Recall that a Montesinos knot or link $M((\alpha_1, \beta_1), \dots, (\alpha_t, \beta_t))$ is defined to be a knot or link connecting t rational tangles of slope from $r_1 = \frac{\beta_1}{\alpha_1}$ through $r_t = \frac{\beta_t}{\alpha_t}$ as indicated in Figure 1 (α_i and β_i are coprime integers). It is well-known that if $t \leq 2$, it is a 2-bridge knot or link. For a reference, see [2] and [3].

In [10], the author studied 2-adjacency relation between Montesinos knots and the trivial knot. In the current short note, we show that for every 2-bridge knot K , there is an infinite family of Montesinos knots $\{K_q\}$ such that each K_q is 2-adjacent to K , and we also present a new example of 2-adjacency relation between 2-bridge knots.

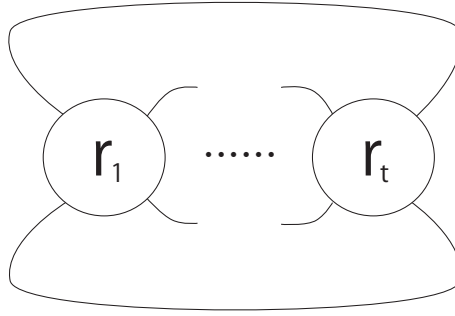
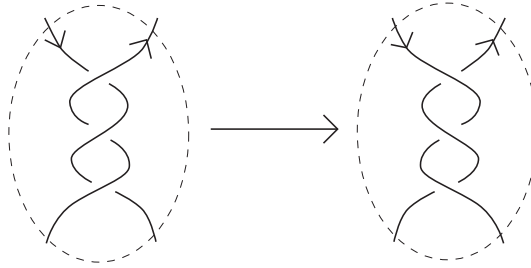


Figure 1: Montesinos knots.

Figure 2: A generalized crossing change of order -3 .

2 Main Theorem

Let K be an oriented knot, link, or tangle in S^3 . A crossing disk for K is an embedded disk $D \subset S^3$ such that K intersects $\text{int}(D)$ twice with zero algebraic intersection number. The boundary of a crossing disk is called a crossing circle. Then, a *generalized crossing change of order $d \in \mathbb{Z}$* on K is achieved by $-\frac{1}{d}$ -Dehn surgery on a crossing circle and results in introducing d full twists to K at the crossing disk (see Figure 2). A *generalized crossing* of order d on a diagram of K is a set C of d twist crossings on two strings that inherit opposite orientations of K . If K' is obtained from K by switching all the crossings in C simultaneously, then K' is the result of a generalized crossing change of order d on K . Note that if $|d| = 1$, K and K' differ by an ordinary crossing change while if $d = 0$, then $K = K'$. Throughout this paper, we assume that $d \neq 0$.

Recall that K is 2-adjacent to K' , if there are two generalized crossings of order d_1 and d_2 in a diagram of K such that correspondingly, both each generalized crossing change and simultaneous generalized crossing changes yield a diagram of K' .

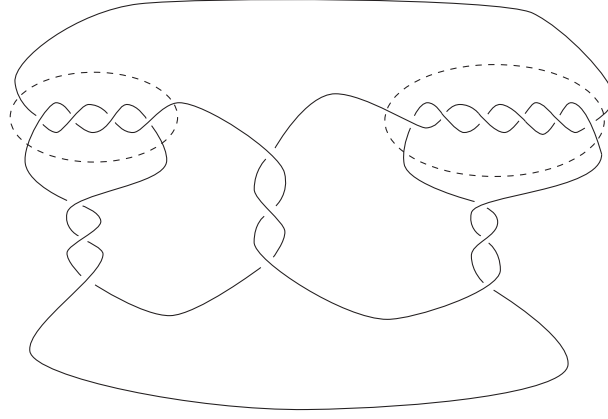


Figure 3: $MK_{(q,d_1,d_2,\varepsilon_1,\varepsilon_2)} = M((2d_1q + \varepsilon_1, 2d_1), (q, -1), (2d_2q + \varepsilon_2, 2d_2))$, where $q = 3, d_1 = 2, d_2 = -3$ and $\varepsilon_1 = \varepsilon_2 = +1$.

Example 1 ([10]). For an integer q , non-zero integers d_1, d_2 and $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$, let $MK_{(q,d_1,d_2,\varepsilon_1,\varepsilon_2)}$ be the Montesinos knot $M((2d_1q + \varepsilon_1, 2d_1), (q, -1), (2d_2q + \varepsilon_2, 2d_2))$ (see Figure 3). It is easily observed that the generalized crossings of order $\varepsilon_1 d_1$ and $\varepsilon_2 d_2$ in the dotted circles for $MK_{(q,d_1,d_2,\varepsilon_1,\varepsilon_2)}$ as shown in Figure 3 make $MK_{(q,d_1,d_2,\varepsilon_1,\varepsilon_2)}$ 2-adjacent to the trivial knot.

Lemma 2. Suppose $r = \frac{2dn^2}{2dmn + \varepsilon}$ for some coprime integers m, n , a non-zero integer d and $\varepsilon = \pm 1$. Further, for $\frac{n}{m} = [a_0, a_1, \dots, a_k] = a_0 + \frac{1}{a_1 + \dots + \frac{1}{a_k}}$, if k is odd (resp. even), let $\epsilon = 1$ (resp. -1). Then, a rational tangle of slope $\frac{0}{1}$ is obtained from a rational tangle of slope r by a generalized crossing change of order $\epsilon \varepsilon d$.

Proof of Lemma 2. The denominator and the numerator of the rational tangle of slope r are a 2-bridge knot and a 2-bridge link with unknotting and unlinking number one by a generalized crossing change, respectively. Then, the statement is contained in the works of Kanenobu–Murakami [6] and Kohn [7]. See, also [8]. In fact, if $\frac{n}{m} = [a_0, \dots, a_k]$, then we have $\frac{2dn^2}{2dmn + \varepsilon} = [a_0, \dots, a_k, 2\varepsilon d, -a_k, \dots, -a_0]$ if k is odd or $[a_0, \dots, a_k, -2\varepsilon d, -a_k, \dots, -a_0]$ if k is even.

We also need the following lemma by Ernst–Sumners [4].

Lemma 3 ([4, Lemma 2·1]). A Montesinos knot or link $M((\alpha_1, \beta_1), (\alpha_2, \beta_2))$ is

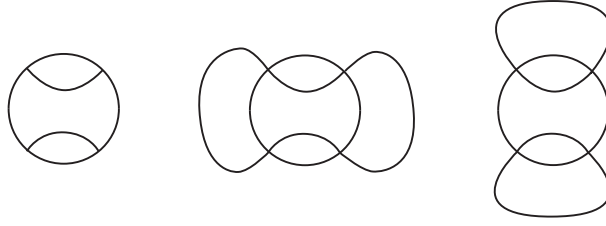


Figure 4: The rational tangle of slope $\frac{0}{1}$, its denominator and numerator.

isotopic to a 2-bridge knot or link $M((x\alpha_2 + y\beta_2, \alpha_1\beta_2 + \beta_1\alpha_2))$, where x and y are any integers such that $\alpha_1x - \beta_1y = 1$.

We generalize Example 1 as follows.

Theorem 4. For coprime integers m, n , non-zero integers d_1, d_2 and $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$, a Montesinos knot or link $M((2d_1mn + \varepsilon_1, 2d_1n^2), (m, -n), (2d_2mn + \varepsilon_2, 2d_2n^2))$ is 2-adjacent to a 2-bridge knot or link $M((m, -n))$ by two generalized crossing changes of order $\varepsilon_1\varepsilon_1d_1$ and $\varepsilon_2\varepsilon_2d_2$, where $\varepsilon_i = \pm 1$ are as in Lemma 2.

Corollary 5. For every 2-bridge knot K and fixed non-zero integers d_1, d_2 , there is an infinite family of Montesinos knots $\{K_q\}$ such that each K_q is 2-adjacent to K by two generalized crossing changes of order d_1 and d_2 .

Proof of Corollary 5. For an integer q , $M((m+nq, -n))$ is isotopic to $M((m, -n))$. Hence, unless $n = 0$, we can easily construct a family $\{K_q\}$ by Theorem 4 as desired.

By putting $m = 1$, we also have the following new example of 2-adjacency relation between 2-bridge knots or links.

Corollary 6. A 2-bridge knot or link $M((2d_1n + \varepsilon_1, 2d_1n^2), (1, -n), (2d_2n + \varepsilon_2, 2d_2n^2))$ is 2-adjacent to a 2-bridge knot or link $M((1, -n))$ by two generalized crossing changes of order $\varepsilon_1\varepsilon_1d_1$ and $\varepsilon_2\varepsilon_2d_2$ (for the case $d_1 = d_2 = \varepsilon_1 = \varepsilon_2 = 1$ and $n = 3$, see Figure 5).

Remark 7. In [11], we proved that if a 2-bridge knot or link $M((m, n))$ is 2-adjacent to a 2-bridge knot or link $M((m', n'))$, then n is divisible by n' , and conjectured that if a 2-bridge knot K is 2-adjacent to a 2-bridge knot $K' (\neq K)$ by two generalized crossing changes of order d_1, d_2 , then $K = M((\alpha, \beta))$, where $\frac{\alpha}{\beta} = [2d_1, 2d_2]$

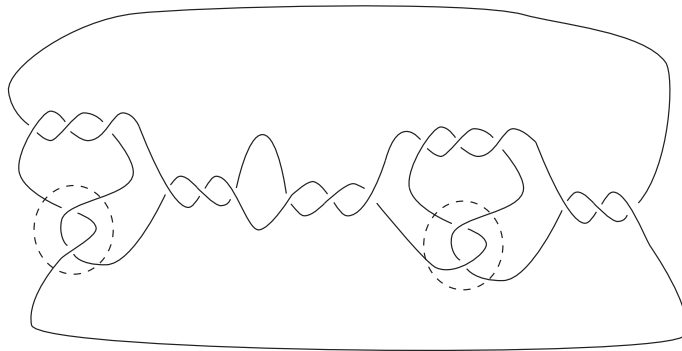


Figure 5: The 2-bridge knot $M((7, 18), (1, -3), (7, 18))$ is 2-adjacent to the 2-bridge knot $M((1, -3))$, here $\frac{18}{7} = [3, -2, -3]$ ($\epsilon_1 = \epsilon_2 = -1$).

or $[2d_2, 2d_1]$ in Conway's notation, and K' is the trivial knot. Therefore, Corollary 6 gives a counterexample to the conjecture of 2-adjacency relation between 2-bridge knots.

Proof of Theorem 4. By Lemma 2, it is easy to see that $M((m, -n))$ is obtained from $M((2d_1mn + \epsilon_1, 2d_1n^2), (m, -n), (2d_2mn + \epsilon_2, 2d_2n^2))$ by simultaneous generalized crossing changes of order $\epsilon_1\epsilon_1d_1$ and $\epsilon_2\epsilon_2d_2$. Thus, it is sufficient to show that $M((2d_1mn + \epsilon_1, 2d_1n^2), (m, -n))$ and $M((m, -n), (2d_2mn + \epsilon_2, 2d_2n^2))$ are isotopic to $M((m, -n))$, respectively (Figure 6). If we put $x = -2d_1mn + \epsilon_1$ and $y = -2d_1m^2$, then we have $(2d_1mn + \epsilon_1)x - (2d_1n^2)y = 1$ and $xm + y(-n) = \epsilon_1m$. Therefore, by Lemma 3, we can conclude that $M((2d_1mn + \epsilon_1, 2d_1n^2), (m, -n))$ is isotopic to $M((\epsilon_1m, -\epsilon_1n)) = M((m, -n))$. The proof of another isotopic case is quite similar. This completes the proof.

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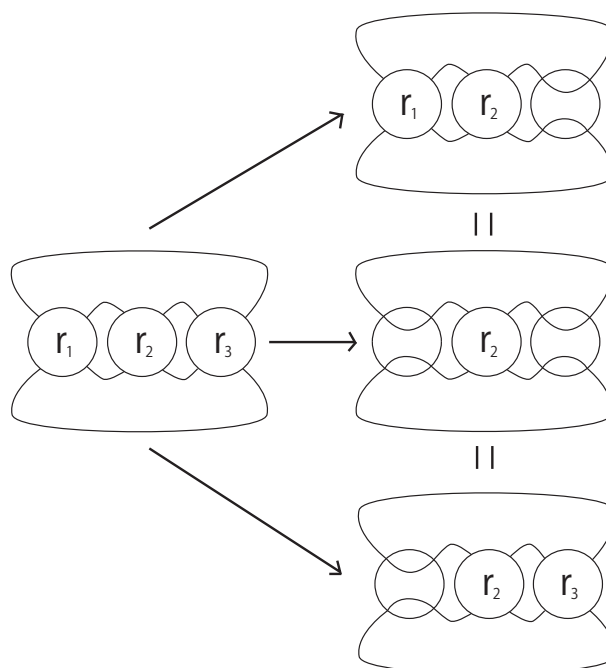


Figure 6: A Montesinos knot or link 2-adjacent to a 2-bridge knot or link.

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