

# On 2-adjacency Relation of Links

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## Abstract

In this note, we will give some examples of links having 2-adjacency relation and we also report a recent study of 2-bridge links with 2-adjacency relation by the author.

## 1 Introduction

Let  $K$  and  $K'$  be links in  $S^3$  with the same number of the components.  $K$  is called  $n$ -adjacent to  $K'$  for some  $n \in \mathbf{N}$ , if  $K$  admits a diagram containing  $n$  crossings such that changing any  $0 < m \leq n$  of them yields a diagram of  $K'$  ([1]). We will write  $K \xrightarrow{n} K'$ . Let  $O$  denote a trivial link. If  $K \xrightarrow{n} O$ ,  $K$  is also said to be *strongly*  $(n-1)$ -trivial ([4]). By definition,  $K \xrightarrow{n} K'$  implies that  $K$  is  $n$ -similar to  $K'$  ([14]). Thus, if  $K \xrightarrow{n} K'$ , finite type invariants of  $K$  and  $K'$  of orders  $< n$  agree.

Remark that  $K \xrightarrow{n} K'$  implies  $K \xrightarrow{n'} K'$  for all  $0 < n' \leq n$  and that  $K \xrightarrow{1} K'$  means the *Gordian distance* between  $K$  and  $K'$  is at most one (cf. [3], [15]). Therefore, we are particularly interested in the case  $n \geq 2$ .

In this note, we will give some examples of links having essential 2-adjacency relation and we also report a recent study of 2-bridge links with 2-adjacency relation by the author.

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## 2 Examples

The trefoil knot and the figure-eight knot are 2-adjacent to the trivial knot (Figure 1). These are the first non-trivial examples

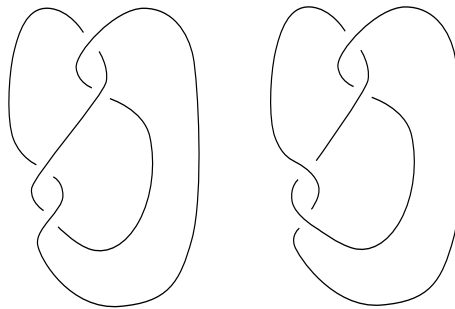


Figure 1:  $3_1$  and  $4_1$ .

of 2-adjacency relation of links. Since these are *fibered* knots, the following theorem of Kalfagianni in [5] shows the converse of the relations are not true.

**Theorem 1 (Kalfagianni, [5]).** Suppose  $K \xrightarrow{2} K'$  and  $K'$  is a fibered knot. Then  $g(K) > g(K')$  or  $K = K'$ , where  $g(\cdot)$  is the Seifert genus.

In [4], Howards–Luecke constructed a strongly  $n$ -trivial non-trivial knot for every  $n$  and in [1], Askitas–Kalfagianni gave a general construction of a strongly  $n$ -trivial knot. The following are examples of the pair of knots  $K, K'$  in  $S^3$  such that (i)  $K \xrightarrow{2} K'$ , (ii)  $K$  and  $K'$  are non-trivial, (iii)  $K$  and  $K'$  are prime, and (iv)  $K \neq K'$ .

**Example 2.** Let  $K$  be a knot as illustrated in Figure 2 and  $K'$  be the (right-handed) trefoil knot. The crossing changes in the dotted circles show that  $K \xrightarrow{2} K'$  (in fact,  $K \xrightarrow{3} K'$ ). Since  $K$  is a *satellite* knot, we can conclude that  $K$  is non-trivial and  $K \neq K'$ .  $K$  is a *periodic* knot of period 3. Since the factor knot is unknotted (Figure 3), it follows that  $K$  is prime by a standard immersed 2-sphere argument.

**Example 3.** Let  $K$  be a knot as illustrated in Figure 4 and  $K'$  be the  $(-2, 3, 5)$ -*pretzel* knot. The crossing changes in the dotted circles show that  $K \xrightarrow{2} K'$ . Again since  $K$  is a satellite knot, we can conclude that  $K$  is non-trivial and  $K \neq K'$ . By a *prime tangle* argument due to Nakanishi ([8, Chapter 3]), it can be proved that  $K$  is prime.

**Example 4.** Let  $K$  be a knot as illustrated in Figure 5. Then,

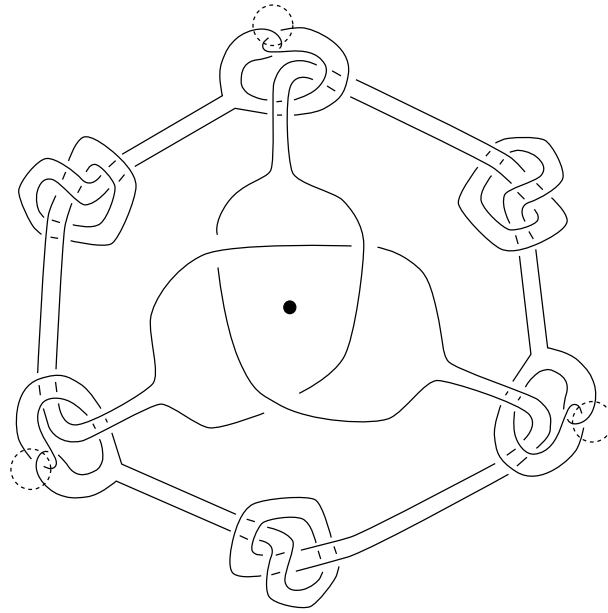


Figure 2: A periodic knot 3-adjacent to the trefoil knot.

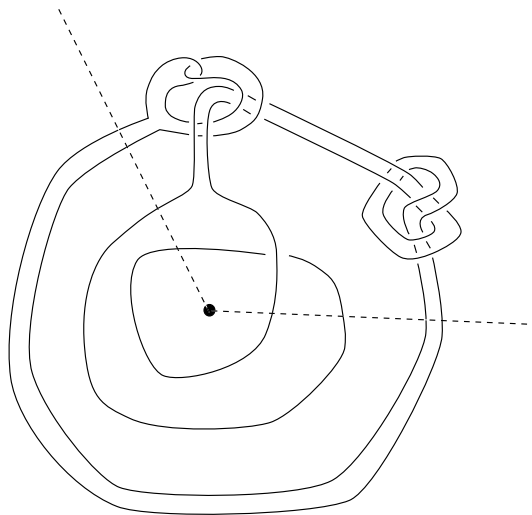


Figure 3: The factor knot is unknotted.

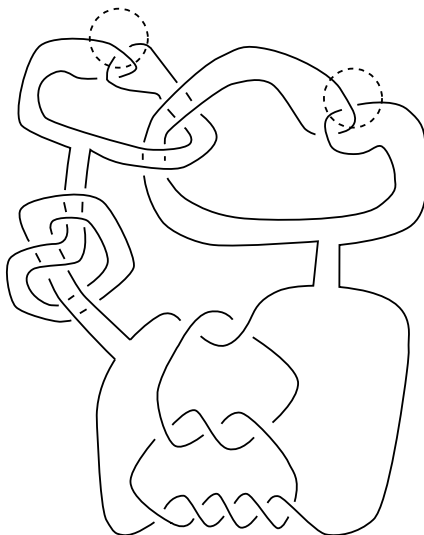


Figure 4: A knot 2-adjacent to the  $(-2, 3, 5)$ -pretzel knot.

the crossing changes in the dotted circles show that  $K$  is 2-adjacent to a *twist* knot. Again since  $K$  is a satellite knot, we can conclude that  $K$  is non-trivial and  $K \neq K'$ . By the crossing of the asterisk, it is easily seen that the *unknotting number* of  $K$  is one. Hence  $K$  is prime by the Scharlemann's theorem in [13].

**Remark 5.** In Examples 2,3 and 4, by replacing the satellite parts as in Figure 6, we may have another examples of  $K$ . Then in Examples 2 and 4, non-triviality of  $K$  are also given by Theorem 1 and Theorem 7 in the next section. But the proof of  $K \neq K'$  may need some calculations of an algebraic knot invariant such as *Conway polynomial*.

The following conjecture is very plausible by the works of Kalfagianni–Lin ([5], [6]).

**Conjecture 6 (cf. [6], [16]).** Suppose  $K \xrightarrow{2} K'$ . Then we have the following:

- (i) If  $K \neq K'$ , then  $g(K) > g(K')$ .
- (ii) If  $K = K'$ , then all the crossing needed in the definition of 2-adjacency of  $K$  is *nugatory*, namely the corresponding crossing circle bounds a disk disjoint from  $K$ .

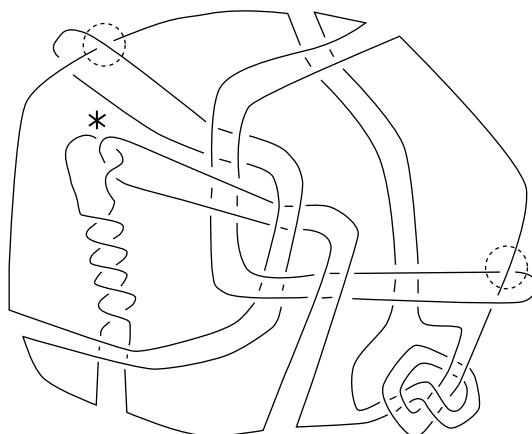


Figure 5: An unknotting number one knot 2-adjacent to a twist knot.

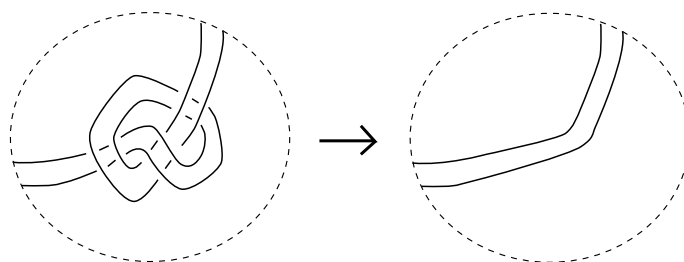


Figure 6: A replacement of a 2-string tangle.

### 3 The case of 2-bridge links

Generalizing the Kanenobu–Murakami and Kohn’s results of 2-bridge knots and links with unknotting number one ([7], [9]), Darcy–Sumners and the author completely determined 1-adjacency relation of 2-bridge knots and links ([3], [15]). Then we should remark that the main ingredient of the proofs are the *Montesinos trick* ([11]) and the Culler–Gordon–Luecke–Shalen’s *cyclic surgery theorem* ([2]). Recently, by using the same techniques the author analyzed 2-bridge knots and links with 2-adjacency relation. In this section, we report the study.

Let  $S(p, q)$  be a 2-bridge knot or link in  $S^3$  whose 2-fold branched cover is the lens space  $L(p, q)$ , where  $p, q$  are relatively prime integers.  $S(p, q)$  is a 2-component link for an even  $p$  and a knot for an odd  $p$ . In particular  $S(0, 1)$  is the 2-component trivial link and  $S(1, 0)$  is the trivial knot.

**Theorem 7 ([18]).** Suppose  $S(p, q)$  is 2-adjacent to  $S(r, s)$ . Then we have the following:

- (i)  $p$  is factorized by  $r$ , and
- (ii) if  $p = r$ ,  $S(p, q) = S(r, s)$ .

**Corollary 8 ([18]).** (i) Suppose the trivial knot is 2-adjacent to  $S(p, q)$ . Then  $S(p, q)$  is the trivial knot  $S(1, 0)$ .

(ii) Suppose the 2-component trivial link is 2-adjacent to  $S(p, q)$ . Then  $S(p, q)$  is the trivial link  $S(0, 1)$ .

(iii) Suppose  $S(p, q)$  is 2-adjacent to  $S(p, -q)$ . Then  $S(p, q) = S(p, -q)$ , that is to say,  $S(p, q)$  is amphicheiral.

(iv) For a prime integer  $p$ , suppose  $S(p, q)$  is 2-adjacent to  $S(r, s)$ . Then  $S(p, q) = S(r, s)$  or  $S(r, s) = S(1, 0)$ .

**Theorem 9 ([17]).** A 2-bridge knot is 2-adjacent to the trivial knot if and only if it is the trivial knot or the trefoil knot or the figure-eight knot.

Notice that Theorem 9 is not included in Theorem 7. In fact, the proof of Theorem 9 essentially needs a Miyazaki–Motegi’s theorem on *twisting operation* of *torus* knots ([10], [12]). See [17].

Finally, here are some problems.

**Problem 10.** For any prime, non-trivial knot  $K'$  in  $S^3$ , is there always a prime knot  $K$  such that  $K$  is 2-adjacent to  $K'$  and  $K \neq K'$ ?

**Problem 11.** Let  $K$  and  $K'$  be 2-bridge knots. Suppose  $K$  is 2-adjacent to  $K'$  and  $K \neq K'$ . Then, is the pair  $(K, K')$  (the trefoil knot, unknot) or (the figure-eight knot, unknot)?

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